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Spaces and maps approximation and fixed points ☆

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Abstract

A general approximation property for topological spaces is studied in relation with fixed point theory for set-valued maps. A particular instance of this property is the admissibility in the sense of Klee. Examples of “convex” sets of topological spaces equipped with a local topological convexity structure as well as general classes of approximative neighborhood retracts are shown to have this approximation property. A general topological principle on the preservation of the fixed point property under this space approximation is proved. It allows the passage from basic classes of spaces to more elaborate ones for general classes of nonconvex set-valued maps. © 2000 Elsevier Science B.V. All rights reserved.

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1. Introduction

The Brouwer fixed point theorem [7] is one of the deepest existence results in Analysis. It guarantees the existence of a fixed point for a continuous transformation of a closed ball in a Euclidean space. It has been extended to compact convex subsets of possibly infinite dimensional locally convex spaces by Schauder [36]. A significant generalization to continuous transformations with relatively compact range (called compact transformations) of convex subsets of locally convex spaces was

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given by Tychonoff [39]. The replacement of the compactness of the domain by that of the transformation played a decisive role in the solvability of nonlinear equations where the domains are convex subsets of functional spaces which are either unbounded or have nonempty interiors (see [14] for a variety of examples).

A wide new field of applications was opened by the extensions of the Brouwer, Schauder, and Tychonoff theorems to upper semicontinuous set-valued maps with closed convex values by Kakutani [25], Ky Fan [15] and Himmelberg [20], respectively. We refer to Park's survey paper [31] for a detailed historical account of Brouwer's theorem and its off-springs.

It is crucial to point out that concepts of approximation play a key role in both passages:

single-valued mappings $\rightarrow$ set-valued mappings

and

compact domains $\rightarrow$ compact maps.

The first important step in extending Brouwer's theorem to an upper semicontinuous set-valued map Φ with closed convex values is to show that the graph of Φ can be approximated by the graph of single-valued continuous mappings. The Brouwer fixed point theorem (applied to those approximations) provides a net of approximate fixed points of Φ which, due to the compactness of the domain, has a cluster point. Since Φ has closed graph, this cluster point must be a fixed point of Φ. This approximation property has been used by several authors and extensively studied in [1,3].

On the other hand, the passage from compact domains to compact maps can be addressed by observing that compact subsets of convex sets in locally convex spaces can be approximated by convex finite polyhedra contained in the underlying convex domain (see Example 2.2 below) via the so-called Schauder projection. Kakutani's theorem applied to the restrictions of the map in question to those convex finite polyhedra provides a net of approximate fixed points which converges, due to the compactness and the closedeness of the map, to a fixed point.

This space approximation property was elegantly used by Klee [26] who called it *space admissibility*.

A purely topological notion of space approximation which contains the admissibility of Klee and accommodates recent topological notions of convexity is defined in Section 2. We show there that a large class of approximative neighborhood extension spaces have this approximation property thus generalizing Klee's results.

From these observations we derive a key fixed point principle for set-valued maps (Theorem 4.3) which not only allows the passage from basic domains (e.g. closed balls in finite dimensions or convex finite polyhedra) to more elaborate ones (absolute retracts, absolute neighborhood retracts, "topologically convex" sets, etc.) but also permits the shifting of compactness (or substitutes to compactness) from domains to maps.

Our results unify a number of recent generalizations of the Kakutani–Fan–Himmelberg fixed point theorem and shed some light on the role of topology in fixed point theory for set-valued maps. The reader is referred to [4,24,30,32,33] for related results and applications to multivalued variational inequalities.

For the reader's convenience, essential topological notions on topological convexities, retracts, and extension spaces are discussed in Appendix A.

2. Space approximation

In this section we define an approximation property that plays a crucial role in fixed point theory. We show that compact subsets of topologically convex sets in spaces with abstract convexity structures as well as compact subspaces of approximative neighborhood extension spaces can be approximated by finite polyhedra.

We start this section by fixing some basic notations and terminology:

- The *identity mapping* on a set X is denoted by id_X.
- By *space*, we mean Hausdorff topological space. The closure of a subspace A of a space X is as usually denoted by $\bar{A}$.
- Given a subset K of a space X, $\mathrm{Cov}_X(K)$ denotes the collection of all covers of K by open subsets of X; $\mathrm{Cov}(X) := \mathrm{Cov}_X(X)$. We write $\bigcup \omega := \bigcup_{W \in \omega} W$. Given two covers $\omega, \omega' \in \mathrm{Cov}_X(K)$, $\omega' \succcurlyeq \omega$ means that ω' is a refinement of ω. The *star* of a subset $A \subset X$ with respect to a cover $\omega \in \mathrm{Cov}(X)$ is the set $\bigcup \{W \in \omega: A \cap W \neq \emptyset\}$. A cover ω' is a *barycentric refinement* of a cover ω if the cover $\{\mathrm{St}(x, \omega'): x \in X\}$ refines ω. ω' is a *star refinement* of ω if the cover $\{\mathrm{St}(W'\omega'): W' \in \omega'\}$ refines ω.
- Given a uniform space $(X, \mathscr{U})$, $x \in X$, $A \subset X$, and $U \in \mathscr{U}$, $U[x] := \{x' \in X: (x, x') \in U\}$ and $U[A] := \bigcup_{x \in A} U[x]$.
- Given a set X, a space Y, and $\omega \in \mathrm{Cov}(Y)$, two single-valued maps $f, g: X \to Y$ are said to be:
  - ω-*near* if for each $x \in X$, $\{f(x), g(x)\} \subset W$ for some member W of ω.
  - ω-*homotopic* if there exists a homotopy $h: X \times [0,1] \to Y$ joining f and g and such that for each $x \in X$, $h(\{x\} \times [0,1]) \subset W$ for some member W of ω (such a homotopy is called an ω-homotopy between f and g).
- The class of *continuous* single-valued mappings from a space X into a space Y is denoted by $\mathbf{c}(X, Y)$.
- A topological space P is called a *polyhedron* if there exists a simplicial complex K such that the space $|K|$ of K is homeomorphic to P. A triangulation of a polyhedron P is a pair $T = (K, \tau)$ where K is a simplicial complex and $\tau: |K| \to P$ is a homeomorphism (see [12,8]). The space $|K|$ is the *geometric realization* of P; it has vertices at the unit points in a linear space equipped with the topology induced by the Euclidean topology of its finite-dimensional flats. We often do not distinguish between P and $|K|$. The set of all vertices of (the geometric realization of) a polyhedron P is denoted by P^0. A polyhedron is *finite* if P^0 is a finite set (such a polyhedron is a compact space). A polyhedron is *locally finite* if each vertex of its geometric realization belongs to a finite number of simplexes.
- Let $\omega = \{W_i: i \in I\} \in \mathrm{Cov}(X)$ be any cover of a space X. Define the *nerve* of ω as the complex P consisting of all simplexes $\sigma = (u_0, \ldots, u_p)$ for which $W_{i_0} \cap \cdots \cap W_{i_p} \neq \emptyset$. The *geometric nerve* of ω is the geometric realization $|N(\omega)|$ of P.
- A *finite convex polyhedron* can be viewed as the convex hull, in a linear space, of a finite set of vectors.

The approximation property defined and studied below is motivated by a result of Klee [26] on the existence of arbitrarily small continuous displacements of compact sets into finite polyhedra.

The reader is referred to Appendix A for basic facts on (neighborhood) retracts and extension spaces and to Granas [19] for a detailed exposition, along the same lines of thought, on the Lefschetz theory for single-valued maps.

Definition 2.1. Let $\mathscr{K}, \mathscr{P}$ be two classes of spaces. A space X is said to have the $(\mathscr{K};\mathscr{P})$-approximation property (respectively the $(\mathscr{K};\mathscr{P})_H$-approximation property) written $X \in \mathscr{A}(\mathscr{K};\mathscr{P})$ ($X \in \mathscr{A}_H(\mathscr{K};\mathscr{P})$ respectively), if and only if

$$\begin{cases} \forall K \in \mathscr{K} \text{ with } K \underset{\text{closed}}{\subseteq} X, \forall \omega \in \operatorname{Cov}_X(K), \\ \exists \omega' \in \operatorname{Cov}_X(K), \omega' \succcurlyeq \omega, \exists P \in \mathscr{P}, \\ \exists \text{ a pair of continuous functions } s: \bigcup \omega' \to P, \\ r: P \to \bigcup \omega' \text{ such that } r \circ s \text{ and the identity } \mathrm{id}_{\cup \omega'} \\ \text{are } \omega\text{-near } (\omega\text{-homotopic respectively}). \end{cases} \tag{1}$$

It is clear that $\mathscr{A}_H(\mathscr{K};\mathscr{P}) \subset \mathscr{A}(\mathscr{K};\mathscr{P})$.

Convex subsets of locally convex spaces have these approximation properties.

Example 2.2. Let $\mathscr{K}$ be the class of compact spaces, $\mathscr{P}$ the class of finite polyhedra, and let X be a non-empty convex subset of a locally convex topological vector space E. Then

$$X \in \mathscr{A}_H(\mathscr{K};\mathscr{P}).$$

Proof. Let K be compact subset of X and $\omega \in \operatorname{Cov}_X(K)$. Let U a convex symmetric open neighborhood of the origin in E such that $\omega' := \{(x_i + U) \cap X\}_{i=1}^n \in \operatorname{cov}_X(K)$ is a finite refinement of ω. The Schauder projection associated to U, $s : \bigcup \omega' \to X$ is defined by

$$s(x) := \frac{1}{\sum_{i=1}^n \mu_i(x)} \sum_{i=1}^n \mu_i(x) x_i,$$

where $\mu_i(x) := \max\{0, 1 - p_U(x - x_i)\}$, p_U being the Minkowski functional associated to U. It verifies

$$\forall x \in \bigcup \omega', s(x) - x \in U, \quad s(x) \in P,$$

where P is a finite polyhedron, copy of the geometric realization of the cover ω'. If r denotes the inclusion $P \hookrightarrow X$, then $s \circ r$ and $\mathrm{id}_{\cup\omega'}$ are ω-near, i.e., $X \in \mathscr{A}(\mathscr{K};\mathscr{P})$. Since U is convex, for each $x \in \bigcup \omega'$, the line segment joining $s(x)$ to x belongs to $(x + U) \cap X$. Hence $s \circ r$ and $\mathrm{id}_{\cup\omega'}$ are ω-homotopic through the linear homotopy $h(x,t) := t((s \circ r)(x)) + (1-t)x$, $t \in [0,1]$. Therefore, $X \in \mathscr{A}_H(\mathscr{K};\mathscr{P})$. $\square$

The following observation is quite important for the sequel.

Remark 2.3. The finite polyhedron P is not necessarily convex. The refinement ω' can be chosen so that $P \subset \bigcup \omega$ (to be precise, $P \subset \bigcup_{i=1}^n \{(x_i + 2U) \cap X\}$). Obviously, P is contained in the convex hull of the finite set $\{x_1, \ldots, x_n\}$, so that in effect, convex subsets of locally convex spaces are in the smaller class $\mathscr{A}$ (compact spaces; convex finite polyhedra). Spaces in this class are known as being admissible in the sense of Klee. This approximation property of compact subsets of convex sets was extended by Klee [26] to some nonlocally convex spaces (e.g., ℓ^p, $0 < p < 1$).

We shall prove next that "convex" subsets of spaces equipped with some abstract topological convexity are also admissible in the sense of Klee.

Proposition 2.4. *Let $\mathscr{K}$ be the class of compact spaces and $\mathscr{P}$ the class of convex finite polyhedra. Assume that X is a nonempty $\mathscr{C}$-convex subset of a locally $\mathscr{C}$-convex space E (see Definition A.6) where $\mathscr{C}$ is:*

(i) *a convexity structure in the sense of Horvath (see Example A.3), or*
(ii) *a B'-convexity (see Example A.5), or*
(iii) *an L-convexity (see Definition A.2).*

Then

$$X \in \mathscr{A}(\mathscr{K}; \mathscr{P}).$$

If in (i) *and* (ii) *the space E is in addition metrizable, then $X \in \mathscr{A}_H(\mathscr{K}; \mathscr{P})$.*

Proof. We only prove (i); the proof of (ii) and (iii) being simpler are left to the reader. We start by showing that $X \in \mathscr{A}(\mathscr{K}; \mathscr{P})$.

Let K be a compact subset of X and let $\omega \in \mathrm{Cov}_X(K)$. The cover ω has a suitable uniform refinement ω' of the form $\{V[y_i] \cap X\colon y_i \in K, i = 0, \ldots, n\}$ of open subsets of X, where $V \in \mathscr{V}$, the uniform structure of E.

Let κ be the Kuratowski barycentric mapping that transforms the set $K_V := \bigcup_{i=0}^n V[y_i]$ into a subset of the closure $\bar{\sigma}$ of the open n-simplex $\sigma = 0, \ldots, n$, (see [27,12,8]). The mapping κ verifies

$$\text{for each simplex} \quad i_0 \cdots i_k,\ \kappa^{-1}(i_0 \cdots i_k) = V[y_{i_0}] \cap \cdots \cap V[y_{i_k}] \setminus \bigcup_{i \neq i_j} V[y_i].$$

We shall now show (with an argument similar to the one used in Theorem 1 in [24]) the existence of a continuous mapping $f : \bar{\sigma} \to X$ such that

$$(f \circ \kappa)(x) \in \Gamma(\{y_i\colon x \in V[y_i]\}) \quad \text{for all } x \in K_V.$$

To do this, denote by $\sigma^{(k)}$ the complex consisting of all k-dimensional faces of σ, $k = 0, \ldots, n$, $\sigma^{(n)} = \sigma$. We construct a finite sequence of mappings $f^{(k)} : \overline{\sigma^{(k)}} \to X$, $k = 0, \ldots, n$ with the property $f^{(k)}(\overline{i_0 \ldots i_k}) \subset \Gamma(\{y_{i_j}\colon j = 0, \ldots, k\})$ for every k-dimensional face $i_0 \ldots i_k$ of σ.

If $k = 0$, let $f^{(0)} : \{0, 1, \ldots, n\} \to X$ be the mapping

$$f^{(0)}(i) = z_i \quad \text{where } z_i \text{ is an arbitrary point in } \Gamma(\{y_i\}),\ i = 0, 1, \ldots, n.$$

This mapping is obviously continuous on the set of vertices $\sigma^{(0)}$ of σ equipped with the discrete topology. Assume that for some $k \in \{1, \ldots, n-1\}$ such a function $f^{(k)}$ has been constructed, and let $i_0 \ldots i_{k+1}$ be an arbitrary $(k+1)$-dimensional face of σ. The mapping $f^{(k)}$ maps the boundary of $i_0 \ldots i_{k+1}$ into the set $\bigcup_{l=0}^{k+1} \Gamma(\{y_{i_j}\colon j = 0, \ldots, k+1,\ j \neq l\})$ which, by monotonicity of the c-structure Γ, is contained in the contractible (thus $(k+1)$-connected) set $\Gamma(\{y_{i_0}, \ldots, y_{i_{k+1}}\})$. Therefore, the restriction of $f^{(k)}$ to the boundary of $i_0 \ldots i_{k+1}$ extends continuously into a partial mapping $f^{(k+1)}_{i_0 \ldots i_{k+1}}$ of the entire simplex $i_0 \dots i_{k+1}$ into the set $\Gamma(\{y_{i_0}, \dots y_{i_{k+1}}\})$. Keeping in mind that $\bar{\sigma}$ is equipped with a CW-complex topology, one can certainly piece together these partial mappings to form a continuous mapping $f^{(k+1)}: \overline{\sigma^{(k+1)}} \to X$ verifying $f^{(k+1)}(\overline{i_0 \dots i_{k+1}}) \subset \Gamma(\{y_{i_j}: j = 0, \dots, k+1\})$ for every $(k+1)$-dimensional face $i_0 \dots i_{k+1}$ of σ. By the property of κ, the mapping $f := f^{(n)}$ has the desired property. It is important to observe that given $x \in K_V$, if $x \in V[y_i]$ then $y_i \in V[x]$ which is a $\mathscr{C}$-convex set. Hence,

$$f(\kappa(x)) \in \Gamma(\{y_i: x \in V[y_i]\}) \subset V[x],$$

that is, the mapping $f \circ \kappa$ and the inclusion $K_V \hookrightarrow X$ are V-near, hence $X \in \mathscr{A}(\mathscr{K}; \mathscr{P})$.

In cases (i) and (ii) and in view of Examples A.11 and A.12, X is an AR, hence an ANR. The entourage V in the begining of the proof can be chosen so that $\omega' := \{V[y_i]: i = 0, \dots, n\}$ verifies Proposition A.14 Thus, $f \circ \kappa$ and the inclusion $K_V \hookrightarrow X$ are ω-homotopic. □

Remark 2.5. (1) Obviously, (iii) is more general than (i) and (ii). The proof of (iii) is however simpler in that the mapping f constructed in the proof is readily provided in the definition of an L-structure (see (12)).

(2) We conjecture that the metrizability of E is not necessary for the inclusion $X \in \mathscr{A}_H(\mathscr{K}; \mathscr{P})$ to hold in cases (i) and (ii).

(3) It would be interesting to determine sufficient conditions on the convexity structure $\mathscr{C}$ for the $\mathscr{C}$-convex sets to be in $\mathscr{A}_H(\mathscr{K}; \mathscr{P})$.

We shall show now that $\mathscr{A}(\mathscr{K}; \mathscr{P})$ includes known classes of spaces "modeled" on $\mathscr{P}$ that have been widely used in topology, in particular in homotopy theory.

Let X, P be two spaces. If there are continuous mappings $s \in \mathbf{c}(X, P)$ and $r \in \mathbf{c}(P, X)$ such that the diagram $\mathrm{id}_X \circlearrowleft X \underset{r}{\overset{s}{\rightleftarrows}} P$ commutes, we say that P *dominates* X. The class of spaces that are dominated by members of a given class $\mathscr{P}$ of spaces is

$$X \in R(\mathscr{P}) \Leftrightarrow X \text{ is dominated by some } P \in \mathscr{P}. \tag{2}$$

Given a cover $\omega \in \mathrm{cov}(X)$, we say that P *ω-dominates* X (*ω-H-dominates* X, respectively) if there are mappings $s \in \mathbf{c}(X, P)$ and $r \in \mathbf{c}(P, X)$ such that $r \circ s$ and id_X are ω-near (ω-homotopic respectively). The classes $D(\mathscr{P})$ and $D_H(\mathscr{P})$ are defined by

$$\begin{array}{c} X \in D(\mathscr{P}) \\ (X \in D_H(\mathscr{P}),\ \text{respectively}) \end{array} \Leftrightarrow \begin{cases} \forall \omega \in \mathrm{cov}(X), \exists P \in \mathscr{P}, \\ \text{such that } P\omega\text{-dominates } X \\ (\omega\text{-H-dominates } X \text{ respectively}). \end{cases} \tag{3}$$

It is clear that for any class $\mathscr{P}$, $\mathscr{P} \subset R(\mathscr{P}) \subset D_H(\mathscr{P}) \subset D(\mathscr{P})$ and that for any class $\mathscr{K}$, $\mathscr{K} \cap \mathscr{A}(\mathscr{K}; \mathscr{P}) \subseteq D(\mathscr{P})$ and $\mathscr{K} \cap \mathscr{A}_H(\mathscr{K}; \mathscr{P}) \subseteq D_H(\mathscr{P})$.

Proposition 2.6. $D(\mathscr{P}) \subset \mathscr{A}(\mathscr{K}; \mathscr{P})$ and $D_H(\mathscr{P}) \subset \mathscr{A}_H(\mathscr{K}; \mathscr{P})$ for any given classes of spaces $\mathscr{P}$, $\mathscr{K}$.

Proof. We only prove the first inclusion, the proof of the second one being identical. Let $X \in D(\mathscr{P})$, $K \in \mathscr{K}$, be a closed subspace of X, and let $\omega \in \mathrm{Cov}_X(K)$. By hypothesis, $\exists P \in \mathscr{P}$, $\exists s \in \mathbf{c}(X, P)$, $\exists r \in \mathbf{c}(P, X)$ such that $r \circ s$ and id_X are $\hat{\omega}$-near, where $\hat{\omega} := \omega \cup \{X \backslash K\}$ is an open cover of X. For any $x \in K$, the member W_x of $\hat{\omega}$ containing $\{x,(r \circ s)(x)\}$ cannot be $X \backslash K$. The open set $U_x := (r \circ s)^{-1}(W_x)$ is a neighborhood of x in X and the family $\omega' := \{W_x \cap U_x\colon x \in K\}$ is an open cover of K that refines ω. The mappings $s|_{\cup\omega'} \circ r$ and $\mathrm{id}_{\cup\omega'}$ are ω'-near. $\square$

The passages $\mathscr{P} \to D(\mathscr{P})$ or $D_H(\mathscr{P})$ preserve certain fundamental topological properties of spaces. As a result, they are used to enlarge basic classes of spaces without altering topological "features". For example, the passage from finite to infinite products of spaces can be described by the domination property (3).

Indeed, given an arbitrary family of spaces $\{X_i\colon i \in I\}$, $X = \prod_{i\in I} X_i$, and given $J \in \mathscr{J} = \{J \subset I\colon J$ is finite$\}$, let $X_J = \prod_{i\in J} X_i$. Then we have:

Proposition 2.7. If X is paracompact and $\mathscr{P} = \{X_J\colon J \in \mathscr{J}\}$, then $X \in D(\mathscr{P})$.

Proof. The proof is identical to that of Proposition 4.1 in [19] where the compact case is treated.

For each $i \in I$, let x_i^0 be a base point in the space X_i and for $J \in \mathscr{J}$, let $\tilde{X}_J \subset X$ be the product $X_J \times (x_i^0)_{i\notin J}$, i.e.,

$$(x_i) \in \tilde{X}_J \Leftrightarrow \begin{cases} x_i \in X_i & \text{if } i \in J, \\ x_i = x_i^0 & \text{if } i \notin J. \end{cases}$$

Clearly, $\tilde{X}_J$ can be identified with X_J.

Consider the subcollection $\mathscr{C}$ of $\mathrm{Cov}(X)$ defined as follows:

$$\omega \in \mathscr{C} \Leftrightarrow \begin{cases} \omega \text{ is a neighborhood-finite cover of } X \text{ by open sets} \\ \text{of the form } \prod_{i\in I} U_i \text{ where, for each } i \in I, \\ U_i \text{ is an open subset of } X_i, \text{ and } U_i = X_i \\ \text{except for at most finitely many indices } i. \end{cases}$$

Since X is paracompact and by the very definition of the Cartesian product topology, it follows that $\mathscr{C}$ is cofinal in $\mathrm{Cov}(X)$. Let $\omega = \{\omega_\lambda\}_{\lambda\in\Lambda} \in \mathscr{C}$ be arbitrary and let Λ_0 be the finite set of those indices λ for which $x_0 \in \omega_\lambda$. By definition, each ω_λ, $\lambda \in \Lambda_0$, is of the form $\prod_{i\in I} U_i^\lambda$ where, for each $i \in I, U_i^\lambda$ is an open subset of X_i, and $U_i^\lambda = X_i$ except for some essential indices forming a finite set $I(\lambda) := \{i \in I\colon U_i^\lambda \neq X_i\}$. The set $J(\omega) := \bigcup_{\lambda\in\Lambda_0} I(\lambda)$ is clearly finite and $i \notin J(\omega) \Leftrightarrow U_i^\lambda = X_i$ for all $\lambda \in \Lambda_0$.

Let $s\colon X \to X_{J(\omega)}$ be the projection and $r : \tilde{X}_{J(\omega)} \to X$ be the natural imbedding. One readily verifies that $r \circ s$ and id_X are ω-near. This completes the proof. $\square$

Properties (1)–(3) can be used to describe the passage from basic types of spaces to more elaborate ones. For instance:

Example 2.8. (i) If $\mathscr{P}$ is the class of all normed spaces, then $R(\mathscr{P})$ is the class AR of absolute retracts.

(ii) If $\mathscr{P}$ is the class of all open subsets of normed spaces, then $R(\mathscr{P})$ is the class ANR of absolute neighborhood retracts.

(iii) If $\mathscr{P}$ is the class of all finite (respectively, locally finite) polyhedra endowed with the CW-topology, then $D_H(\mathscr{P})$ contains the class of compact (respectively, arbitrary) ANRs.

(iv) Examples A.11 and A.12 together with (iii) yield to a refinement of Proposition 2.4: $\mathscr{C}$-convex subsets of locally $\mathscr{C}$-convex metrizable spaces in the sense of Horvath or in the sense of Bielawski are in $D_H(\mathscr{P})$ where $\mathscr{P}$ is the class of all finite polyhedra.

The following remark is important for the sequel:

Remark 2.9. (1) One should keep in mind that in the proof of inclusion (iii) in Example 2.8, for a given $\omega \in \mathrm{Cov}(X)$, the polyhedron P that ω-H-dominates an ANR X is precisely the geometric nerve $|N(\omega)|$ of the cover ω (see [8,12]). (Moreover, the mappings s and r involved in (3) are, respectively, essentially the Kuratowski barycentric mapping κ (as in the proof of Proposition 2.4) and the neighborhood retraction involved in the definition of an ANR.)

(2) The property, for a paracompact space X, of being ω-H-dominated is "transitive" in the following sense:

$$\text{If } X \in D_H(\mathscr{P}) \text{ and } \mathscr{P} \subset D_H(\mathscr{P}'), \text{ then } X \in D_H(\mathscr{P}').$$

Indeed, given $\omega \in \mathrm{Cov}(X)$, let $\omega^* \in \mathrm{Cov}(X)$ be a star refinement of ω and let $P \in \mathscr{P}, s \in \mathbf{c}(X,P), r \in \mathbf{c}(P,X)$ be such that $r \circ s$ and id_X are ω^*-homotopic. Let $\alpha^* = r^{-1}(\omega^*) \in \mathrm{Cov}(P)$ and let $P' \in \mathscr{P}', s' \in \mathbf{c}(P,P'), r' \in \mathbf{c}(P',P)$ be such that $r' \circ s'$ and id_P are α^*-homotopic (through a homotopy $h : P \times [0,1] \to P$). For any $x \in X$, $\{(r' \circ s' \circ s)(x), s(x)\} \subset A^* = r^{-1}(W^*)$, for some $W^* \in \omega^*$, that is $\{(r \circ r' \circ s' \circ s)(x), (r \circ s)(x)\} \subset W^*$. But $\{(r \circ s)(x), x\} \subset W^{*'}$ for some $W^{*'} \in \omega^*$. Thus $\{(r \circ r' \circ s' \circ s)(x), (r \circ s)(x), x\} \subset \mathrm{St}(W^*, \omega^*) \subset W$ for some $W \in \omega$. This means that $P'\omega$-dominates X. Now, observe that $r \circ r' \circ s' \circ s$ and $r \circ s$ are homotopic through the homotopy $r \circ h(s(.),.)$. But $r \circ s$ is homotopic to id_X. Thus, $r \circ r' \circ s' \circ s$ and id_X are homotopic (one can choose ω^* in such a way that the latter homotopy is controlled by ω).

Note that in view of the first part of this remark, if P is an ANR, P' can be chosen to be the nerve $|N(\alpha^*)|$ of the cover α^* and that this nerve is a subpolyhedron of the nerve of ω^* (both nerves having the same vertices).

These remarks lead to the fact

Theorem 2.10. *Assume that $X \in D_H(\mathscr{P})$ where $\mathscr{P}$ is the class of finite polyhedra. If X has non-trivial Euler–Poincaré characteristic $\mathscr{E}(X)$.*[1] *Then, $\forall \omega \in \mathrm{Cov}(X)$, $\exists P \in \mathscr{P}$ such that*: (i) *$P\omega$-H-dominates X, and* (ii) $\mathscr{E}(P) \neq 0$.

Proof. Let $\omega \in \mathrm{Cov}(X)$ be arbitrary but fixed, and let $|N(\omega)|$ be the geometric nerve of ω. By Proposition 5.2, and since $|N(\omega)|$ is an ANR, there exists an open cover $\alpha \in \mathrm{Cov}(|N(\omega)|)$, such that for any metric space Z, any two mappings $f, g \in \mathbf{c}(Z, |N(\omega)|)$ that are α-near are homotopic. Consider now a triangulation τ of $|N(\omega)|$ finer than the cover α. Let us choose the (possibly iterated) star refinement ω^* of ω, the polyhedron P, the cover $\alpha^* = r^{-1}(\omega^*)$ in the proof of Remark 2.9(2) in such a way that $|N(\omega^*)|$ is a subpolyhedron of $(|N(\omega)|, \tau)$, and that the cover $\alpha' = r'r^{-1}(\alpha^*)$ of $|N(\alpha^*)|$

[1] Recall that for a spherical complex X, the Čech cohomology graded linear space $\{H^q(X; \mathbf{Q})\}$ is of finite type. The *Euler–Poincaré characteristic* of X is defined to be the Lefschetz number of the identity mapping, namely, $\mathscr{E}(X)$ is the finite sum $\lambda(\mathrm{id}_X) := \sum_{q \geqslant 0} (-1)^q \beta_q$ where $\beta_q' = \dim H^q(X; \mathbf{Q})$ is the qth-Betti number of X (see [29,37]).

refines the trace of the cover α on $|N(\alpha^*)|$ (keep in mind that $|N(\alpha^*)|$ is a subpolyhedron of $|N(\omega^*)|$, thus $|N(\alpha^*)|$ is a subpolyhedron of $|N(\omega)|$). Denote $|N(\alpha^*)|$ again by P. Obviously, $P\omega$-H-dominates X, and the mappings $\mathrm{id}_P, s'\circ s\circ r\circ r' : P\to P$ are α'-near. Hence, id_P and $s'\circ s\circ r\circ r'$ are homotopic.

By the homotopy invariance of the Lefschetz number, $\mathscr{E}(X)=\lambda(r\circ r'\circ s'\circ s)$ and $\mathscr{E}(P)=\lambda(s'\circ s\circ r\circ r')$. It is well known (e.g. see [19,8]; Lemma 1 III.C) that, when defined for a pair of mappings f and g, the Lefschetz numbers $\lambda(f\circ g)$ and $\lambda(g\circ f)$ are equal. Hence $\mathscr{E}(X)=\lambda(r\circ r'\circ s'\circ s)=\lambda(s'\circ s\circ r\circ r')=\mathscr{E}(P)$. $\square$

We prove now a more general result than Example 2.3(iii); namely that the larger class of approximative neighborhood extension spaces for compacts spaces (see Appendix A for the definition of the classes ES, NES, and ANES) have the approximation property (1).

Theorem 2.11. *Let $\mathscr{K}$ be the class of compact spaces, $\mathscr{P}$ the class of finite polyhedra, and $\mathscr{N}$ the class of normal spaces. Then*:

$$\mathscr{N}\cap ANES(\mathscr{K})\subset\mathscr{A}(\mathscr{K};\mathscr{P}) \quad and \quad \mathscr{N}\cap A_HNES(\mathscr{K})\subset\mathscr{A}_H(\mathscr{K};\mathscr{P}). \tag{4}$$

Proof. We only provide the proof of the first inclusion, the second one being a mere adaptation to the homotopical case. Let $X\in\mathrm{ANES}(\mathscr{K})$ be a normal space, let $K\hookrightarrow X$ be compact, and let $\omega\in\mathrm{Cov}_X(K)$. The Tychonoff imbedding theorem asserts that, as a compact space, K is homeomorphic to a closed subset $\hat{K}$ of a Tychonoff cube T (i.e., a Cartesian product of copies of the unit interval imbedded in a normed space E). Let h be this homeomorphism and let $\omega'\in\mathrm{Cov}_X(K)$ be a barycentric refinement of ω.

By Definition A.8(iv), there exist an open subset V of $T, V\supset\hat{K}$, and $f\in\mathbf{c}(V,X)$ such that h^{-1} and $f|\hat{K}$ in the following diagram:

$$\begin{array}{ccccc} & & X & & \\ & h^{-1}\nearrow & \uparrow f & & \\ \hat{K} & \hookrightarrow & V & \hookrightarrow & T \end{array}$$

are ω'-near.

Due to the compactness of $\hat{K}$, there exist a convex open neighborhood of the origin U in the normed space E containing T and a finite set $\{\hat{x}_i\}_{i=1}^n\subset\hat{K}$ such that $\hat{\omega}:=\{(\hat{x}_i+U)\cap T\}_{i=1}^n\in\mathrm{Cov}_T(\hat{K})$ and $\bigcup_{i=1}^n\{(\hat{x}_i+2U)\cap T\}\subset V$.

The Schauder projection $\pi:\bigcup\hat{\omega}\to P$ onto a finite polyhedron $P\subset\bigcup_{i=1}^n\{(\hat{x}_i+2U)\cap T\}\subset V$ and the identity on $\hat{K}$ are $\hat{\omega}$-near (this follows from Example 2.2 and Remark 2.3).

The Tietze's extension theorem implies that every Tychonoff cube is an extension space for normal spaces, i.e. $T\in\mathrm{ES}(\mathscr{N})$. Moreover, it is known that for a given class of normal spaces $\mathscr{Q}$, every open subspace of an NES($\mathscr{Q}$) space is also NES ($\mathscr{Q}$) (see [19, Théorème 2.2]). Since $\mathrm{ES}(\mathscr{N})\subset\mathrm{NES}(\mathscr{N})$, then $\bigcup\hat{\omega}\in\mathrm{NES}(\mathscr{N})$. Consequently, there exists $\tilde{h}\in\mathbf{c}(\bigcup\omega',\bigcup\hat{\omega})$ such that the diagram

$$\begin{array}{ccccc} & & \bigcup\hat{\omega} & \hookrightarrow & T \\ & h\nearrow & \uparrow\tilde{h} & & \\ K & \hookrightarrow & \bigcup\omega' & \hookrightarrow & X \end{array}$$

commutes. The mappings $s = \pi \circ \tilde{h}$ and $r = f \circ j$ in the following diagram:

$$\begin{array}{ccccc} \bigcup \hat{\omega} & \xrightarrow{\pi} & P & \xhookrightarrow{j} & V \\ \tilde{h} \uparrow & & & & \downarrow f \\ \bigcup \omega' & & \hookrightarrow & & X \end{array}$$

are so that $r \circ s$ and $\mathrm{id}_{\cup\omega'}$ are ω-near, i.e., $X \in \mathscr{A}(\mathscr{K};\mathscr{P})$. □

By Proposition A.15, the class AANR($\mathscr{K}$) of approximative ANRs for compact spaces is precisely $\mathscr{K} \cap$ ANES($\mathscr{K}$) which, by (4), is included in $\mathscr{K} \cap \mathscr{A}(\mathscr{K};\mathscr{P})$ which is in turn contained in $D(\mathscr{P})$; hence

Corollary 2.12. *If $\mathscr{K}$ is the class of compact spaces and $\mathscr{P}$ is the class of finite polyhedra, then*

$$\mathrm{AANR}(\mathscr{K}) \subset D(\mathscr{P}) \quad \textit{and} \quad \mathrm{A}_H\mathrm{ANR}(\mathscr{K}) \subset D_H(\mathscr{P}).$$

Since compact AANRs (in particular, compact ANRs) are in AANR($\mathscr{K}$) for the class $\mathscr{K}$ of compact spaces (see [18]), it follows that

Corollary 2.13. *Every compact* AANR (*in particular, compact* ANR) is in $D(\mathscr{P})$ where $\mathscr{P}$ is the class of finite polyhedra.

3. The class A of approachable set-valued maps

We start with a list of notations used in this section.

- Let **M** be a class of set-valued maps. Following [3], we write:
  - $\mathbf{M}_c := \{\Phi = \Phi_n \circ \cdots \circ \Phi_1\colon \Phi_j \in \mathbf{M}, j = 1,\dots,n\}$.
  - If $\mathscr{K}$ is a class of spaces, $\mathbf{M}^{\mathscr{K}} := \{\Phi \in \mathbf{M}$: the range of Φ is contained in some member of $\mathscr{K}\}$.
  - Given two sets $X, Y, \mathbf{M}(X,Y) := \{\Phi : X \rightrightarrows Y : \Phi \in \mathbf{M}\}; \mathbf{M}(X,X) := \mathbf{M}(X)$.
  - **CL** := class of set-valued maps with closed values.
- The classes of semicontinuous set-valued maps are denoted:
  - **USC** := class of all *upper semicontinuous* set-valued maps with nonempty values between spaces.
  - **USCL** := **USC** $\cap$ **CL**.
  - **USCO** := $\{\Phi \in$ **USC** : Φ has compact values$\}$.
    Clearly, **c** $\subset$ **USCO** $\subset$ **USCL** $\subset$ **USC**.

Definition 3.1. (i) (see [1–3]) Let $(X,\mathscr{U})$ and $(Y,\mathscr{V})$ be two uniform spaces and let $\Phi\colon X \rightrightarrows Y$ be a set-valued map. Given $(U,V) \in \mathscr{U} \times \mathscr{V}$, a single-valued map $s\colon X \to Y$ is said to be a (U,V)-approximative selection of Φ if and only if:

$$\forall x \in X, \exists x' \in U[x] \quad \text{with } s(x) \in V[\Phi(x')]. \tag{5}$$

Denote by $\mathbf{a}(\Phi; U, V) := \{s \in \mathbf{c}(X, Y) : s \text{ is a } (U, V)\text{-approximative selection of } \Phi\}$.

(ii) Φ is said to be approachable if and only if

$$\forall (U, V) \in \mathscr{U} \times \mathscr{V}, \quad \mathbf{a}(\Phi; U, V) \neq \emptyset.$$

Let the class $\mathbf{A}$ of approachable set-valued maps from X into Y be

$$\mathbf{A}(X, Y) := \{\Phi \in \mathbf{usc}(X, Y) \text{: } \Phi \text{ is approachable}\}.$$

The class $\mathbf{A}$ contains several important subclasses. Indeed, consider the following classes of set-valued maps (the topological notions involved in the definitions below are discussed in the appendix):

- $\mathbf{K}(X, Y) := \{\Phi \in \mathbf{USC}(X, Y)$: Φ has nonempty convex values$\}$.
- $\mathbf{C}(X, Y) := \{\Phi \in \mathbf{USC}(X, Y)$: Φ has nonempty $\mathscr{C}$-convex values$\}$, $\mathscr{C}$ being a convexity structure.
- $\mathbf{C}^\infty(X, Y) := \{\Phi \in \mathbf{USCO}(X, Y)$: Φ has nonempty contractible values$\}$.
- $\mathbf{D}(X, Y) := \{\Phi \in \mathbf{USCO}(X, Y)$: Y is a uniform space and the values of Φ have trivial shape in $Y\}$.

Clearly, $\mathbf{K} \subset \mathbf{C}$ and $\mathbf{K} \subset \mathbf{C}^\infty \subset \mathbf{D}$.

Example 3.2. (i) If X is paracompact and Y is a convex subset of a locally convex topological vector space then $\mathbf{K}(X, Y) \subset \mathbf{A}(X, Y)$ [9].

(ii) If X is paracompact and Y is a locally $\mathscr{C}$-convex space, then $\mathbf{C}(X, Y) \subset \mathbf{A}(X, Y)$ in each of the following cases (see [2]):

1. $\mathscr{C}$ is an H-structure;
2. $\mathscr{C}$ is a B-simplicial convexity;
3. $\mathscr{C}$ is an L-structure and X is compact.

(iii) $\mathbf{C}^\infty(P, Y) \subset \mathbf{A}(P, Y)$ provided P is a finite polyhedron [20,3].

(iv) $\mathbf{D}(P, Y) \subset \mathbf{A}(P, Y)$ provided P is a finite polyhedron (see [3,1]; cf. with [18,16]).

Approachability is stable under a number of operations (we refer to [1] for a detailed treatment of these properties). For example, the restriction of an approachable set-valued maps to a compact subset is approachable. Also a Cartesian product of two approachable maps is approachable. The composition product of approachable maps is approachable provided the first space is compact, i.e.,

$$\mathbf{A}_{\mathrm{c}}(X, Y) = \mathbf{A}(X, Y) \text{ provided } X \text{ iscompact}; \tag{6}$$

(it is not known to the author whether this equality holds when the compactness of X is replaced by that of one of the intermediate spaces through which a map in $\mathbf{A}_{\mathrm{c}}$ factorizes).

We shall show next that the classes $\mathbf{C}$ and $\mathbf{D}$ defined above are in the smaller class $\mathbf{A}_H$ defined as follows:

Definition 3.3. A set-valued map $\Phi \in \mathbf{A}(X, Y)$ is said to have property (H) if and only if:

$$\begin{cases} \forall (U, V) \in \mathscr{U} \times \mathscr{V}, \exists (U', V') \in \mathscr{U} \times \mathscr{V}, \forall s_1, s_2 \in \mathbf{a}(\Phi; U', V'), \\ \exists h \in \mathbf{c}(X \times [0,1], Y) \text{ such that } h(.,0) = s_1, h(.,1) = s_2, \text{ and} \\ h(.,t) \in \mathbf{a}(\Phi; U, V), \forall t \in [0,1]. \end{cases} \tag{7}$$

Denote by $\mathbf{A}_H(X,Y)$ the subclass of $\mathbf{A}(X,Y)$ consisting of those set-valued maps having property (H).

We start with a preparatory result.

Let $\mathscr{P}$ be the class of all finite polyhedra and let $(Y,\mathscr{V})$ be a uniform space. A class of set-valued maps $\mathbf{M}$ is said to have the *approximative selection extension property on finite polyhedra* (ASEP ($\mathscr{P}$) for short) if and only if:

$$\begin{cases} \text{(i) } \mathbf{M}(P,Y)\subset \mathbf{A}(P,Y), \text{ and} \\ \text{(ii) } \forall P\in\mathscr{P} \text{ with uniform structure } \mathscr{U}, \forall P_0 \text{ sub-polyhedron of } P \text{ containing the 0} \\ \text{dimensional skeleton of } P, \forall (U,V)\in\mathscr{U}\times\mathscr{V}, \exists (U^0,V^0)\in\mathscr{U}\times\mathscr{V}, \\ \text{such that } \forall s_0\in \mathbf{a}(\Phi|P_0;U^0,V^0), \exists s\in\mathbf{a}(\Phi;U,V), \\ \text{with } s|P_0=s_0. \end{cases} \tag{8}$$

Lemma 3.4. *Let $\mathscr{P}$ be the class of all finite polyehdra, $(Y,\mathscr{V})$ a uniform space, and let $\mathbf{M}$ be an abstract class of set-valued maps satisfying*

$$\forall\Phi\in\mathbf{M}, \forall s\in\mathbf{c}, \textit{ the composition product } \Phi\circ s\in\mathbf{M}. \tag{9}$$

If $\mathbf{M}$ has ASEP($\mathscr{P}$), then $\mathbf{M}(P,Y)\subset\mathbf{A}_H(P,Y), \forall P\in\mathscr{P}$.

Proof. Let $\hat{P} := P\times[0,1]$ and $\hat{P}_0 := (P\times\{0\})\cup(P\times\{1\})$. Clearly, $\hat{P}$ is a finite polyhedron equipped with a product uniformity $\mathscr{U}$ and $\hat{P}_0$ is a subpolyhedron of $\hat{P}$ containing all the vertices of $\hat{P}$. Define $\hat{\Phi}:\hat{P}\rightrightarrows Y$ as

$$\hat{\Phi}(x,t) := \Phi(x), \quad \forall (x,t)\in\hat{P}.$$

The map $\hat{\Phi}$ can be viewed as the composition $\Phi\circ p_1$ where p_1 is the projection of $\hat{P}$ onto P. By hypothesis, $\hat{\Phi}\in\mathbf{M}(\hat{P},Y)\subset\mathbf{A}(\hat{P},Y)$.

For an arbitrarily fixed pair $(U,V)\in\mathscr{U}\times\mathscr{V}$, let $\hat{U}$ be an entourage of the diagonal in $\hat{P}\times\hat{P}$ homeomorphic to a product $U\times[0,1]^2$. By hypothesis, $\exists(\hat{U}^0,V^0)\in\hat{\mathscr{U}}\times\mathscr{V}$, such that any $h_0\in\mathbf{a}(\hat{\Phi}|\hat{P}_0;\hat{U}^0,V^0)$ extends continuously to a mapping $h\in\mathbf{a}(\hat{\Phi};\hat{U},V)$. The set $\hat{U}^0$ contains a copy of a product $U^0\times O^0$ where $U^0\in\mathscr{U}, U^0\subset U$, and O^0 is an entourage of the diagonal in $[0,1]^2$. Now given any $s_1,s_2\in\mathbf{a}(\Phi;U^0,V^0)$, let $h_0:\hat{P}_0\to Y$ be

$$h_0(x,0) := s_1(x) \quad\text{and}\quad h_0(x,1) := s_2(x), \quad \forall x\in P.$$

Clearly, $h_0\in\mathbf{a}(\hat{\Phi}|\hat{P}_0;\hat{U}^0,V^0)$. Let $h\in\mathbf{a}(\hat{\Phi};\hat{U},V)$ be an extension of h_0 to $\hat{P}$. For any pair $(x,t)\in\hat{P}, h(x,t)\in V[\hat{\Phi}(x',t')]$ for some $(x',t')\in\hat{U}[(x,t)]$, i.e., $h(x,t)\in V[\Phi(x')]$ for some $x'\in U[x]$. Thus, $h(.,t)\in\mathbf{a}(\Phi;U,V), \forall t\in[0,1]$ and the proof is complete. □

As an immediate consequence, we have the following refinement of Example 3.2(iii) and (iv):

Proposition 3.5. $\mathbf{D}(P,Y)\subset\mathbf{A}_H(P,Y)$ *provided P is a finite polyhedron and $Y\in\mathrm{NES}(\mathscr{K})$ where $\mathscr{K}$ is the class of compact spaces.*

Proof. In the metrizable case where Y is an ANR, this result is due to [18]. We provide here a shorter proof of this more general case.

In view of Lemma 3.4 and since any composition $\Phi \circ s$ of a set-valued map $\Phi \in \mathbf{D}$ and a continuous single-valued mapping s is also in $\mathbf{D}$, its suffices to show that the class of set-valued maps $\mathbf{D}$ has ASEP($\mathscr{P}$) for the class $\mathscr{P}$ of finite polyhedra.

Let $\Phi \in \mathbf{D}(P, Y)$ where $P \in \mathscr{P}$ and let $\mathscr{U}, \mathscr{V}$ be uniform structures on P and Y, respectively. Since Φ has compact values in an NES($\mathscr{K}$) space, we have (see Section A.1):

$$\begin{cases} \forall x \in P, \forall V \in \mathscr{V}, \exists V_x \in \mathscr{V}, V_x \subset V, \text{ such that} \\ V_x[\Phi(x)] \text{ is contractible in } V[\Phi(x)]; \end{cases}$$

which, due to the compactness of P and the upper semicontinuity of Φ can easily be made uniform in the following sense:

$$\begin{cases} \forall (U, V) \in \mathscr{U} \times \mathscr{V}, \exists (U', V') \in \mathscr{U} \times \mathscr{V}, U' \subset U, V' \subset V, \\ \text{such that } \forall x \in P, \forall n \geqslant 0, \forall s_0 \in \mathbf{c}(\partial\Delta^n, V'[\Phi(U'[x])]), \\ \exists s \in \mathbf{c}(\Delta^n, V[\Phi(U[x])]) \text{ with } s|\partial\Delta^n = s_0. \end{cases} \tag{10}$$

Assume now that P has dimension $n+1$ and let $(U^{n+1}, V^{n+1}) \in \mathscr{U} \times \mathscr{V}$ be arbitrary but fixed. We define a finite sequence $\{(U^s, V^s) \in \mathscr{U} \times \mathscr{V} \colon s = n, n-1, \ldots, 0\}$ as follows: let $(U'^{\,n+1}, V'^{\,n+1})$ be given by (10) and put $V^n = V'^{\,n+1}$; $\forall x \in P$, choose $U''^{\,n+1} \in \mathscr{U}$ such that $\Phi(U''^{\,n+1}[x]) \subset V^n[\Phi(x)]$ and let $U^n \in \mathscr{U}$ be so that $\{U^n[x] \colon x \in P\} \preccurlyeq \{U'^{\,n+1}[x] \cap U''^{\,n+1}[x] \colon x \in P\}$. We then proceed recursively until $s = 0$.

We show that the pair (U^0, V^0) verifies (8).

Let P_0 be a subpolyhedron of P containing all the vertices of P and let $s_0 \in \mathbf{a}(\Phi|P_0; U^0, V^0)$. Assume that for some $0 \leqslant r \leqslant n$ the function s_0 has been extended to an approximative selection $s_r \in \mathbf{a}(\Phi|P_0 \cup P^r; U^r, V^r)$ where P^r is the r-dimensional skeleton of P. It suffices to show that s_r extends to an approximative selection $s_{r+1} \in \mathbf{a}(\Phi|P_0 \cup P^{r+1}; U^{r+1}, V^{r+1})$.

Let (P', P'_0) be a triangulation of (P, P_0) finer than the cover $\{U^0[x] \colon x \in P\}$ of P and let σ be an arbitrary $(r+1)$dimensional simplex of P'^{r+1}. By the choice of P', σ is contained in some open set $U^0[x_\sigma]$, $x_\sigma \in P$, which in turn is contained in $U^r[x_\sigma]$. Thus, for each $x \in \partial\sigma, s_r(x) \in V^r[\Phi(U^r[x] \cap (P_0 \cup P^{r+1}))]$. By the choice of $(U^r, V^r), s^r$ extends to a continuous mapping $s_{r+1} : \sigma \to V^{r+1}[\Phi(U^{r+1}[x] \cap (P_0 \cup P^{r+1}))]$. Hence, the class $\mathbf{D}$ has ASEP($\mathscr{P}$). Lemma 3.4 ends the proof. □

One of the most interesting properties of the class $\mathbf{A}$ is its stability with respect to the domination property (3).

Theorem 3.6. *Let* $\mathbf{M}$ *be an abstract class of set-valued maps with* $\mathbf{M} \subset \mathbf{USC}$ *and satisfying the condition* (9). *If* $\mathscr{P}$ *is a class of compact spaces*, $(X, \mathscr{U})$ *and* $(Y, \mathscr{V})$ *are uniform spaces with* X *paracompact, then*:

(A)

$$\begin{pmatrix} \mathbf{M}(P, Y) \subset \mathscr{A}(P, Y), \\ \forall P \in \mathscr{P} \end{pmatrix} \Rightarrow \begin{pmatrix} \mathbf{M}(X, Y) \subset \mathscr{A}(P, Y) \\ \textit{provided } X \in D(\mathscr{P}) \end{pmatrix},$$

(B)

$$\begin{pmatrix} \mathbf{M}(P, Y) \subset \mathscr{A}_H(P, Y), \\ \forall P \in \mathscr{P} \end{pmatrix} \Rightarrow \begin{pmatrix} \mathbf{M}(X, Y) \subset \mathscr{A}_H(P, Y) \\ \textit{provided } X \in D_H(\mathscr{K}) \end{pmatrix}.$$

Proof. We only prove (B) and refer to Proposition 3.5 in [3] for (A). Given $\Phi \in \mathbf{M}(X,Y)$, let $(U,V) \in \mathscr{U} \times \mathscr{V}$ be arbitrary but fixed. Choose $\hat{U} \in \mathscr{U}$ such that $\hat{U} \circ \hat{U} \subset U$ and let $\omega := \{\hat{U}[x] : x \in X\} \in \mathrm{Cov}(X)$. By (3), there exist $P \in \mathscr{P}$ (with uniform structure $\mathscr{R}$), and mappings $s \in \mathbf{c}(X,P), r \in \mathbf{c}(P,X)$ such that $r \circ s$ and id_X are ω-homotopic. Let $\tilde{h}$ be the ω-homotopy joining these mappings.

Choose $R \in \mathscr{R}$ in such a way that $(r(p), r(p')) \in \hat{U}$ whenever $(p,p') \in R$.

By (9), $\Phi \circ r$ belongs to $\mathbf{M}(P,Y)$ and is approachable as a composition of approachable maps defined on a compact space (see (6) above).

In view of (7) in Definition 3.3, let $U' \in \mathscr{U}, U' \subset \hat{U}, R' \in \mathscr{R}, V' \in \mathscr{V}$ be chosen so that

- the composites $s_1 \circ r$ and $s_2 \circ r \in \mathbf{a}(\Phi \circ r; R', V')$ for some arbitrarily fixed mappings $s_1, s_2 \in \mathbf{a}(\Phi; U', V')$; and
- $s_1 \circ r$ and $s_2 \circ r$ are joined by a homotopy $k : P \times [0,1] \to Y$ such that $k(.,t) \in \mathbf{a}(\Phi \circ r; R, V), \forall t \in [0,1]$.

The homotopy $h_1 : X \times [0,1] \to Y$ defined by

$$h_1(x,t) := k(s(x),t), \quad \forall (x,t) \in X \times [0,1],$$

joins the mappings $s_1 \circ r \circ s$ and $s_2 \circ r \circ s$.

Moreover, $\forall t \in [0,1]$, $h_1(.,t)$ is a (U,V)-approximative selection of Φ. Indeed, $\forall (x,t) \in X \times [0,1]$, $k(s(x),t) \in V[(\Phi \circ r)(R[s(x)])]$, that is, $k(s(x),t) \in V[(\Phi \circ r)(z)]$ for some $z \in R[s(x)]$. By the choice of R, $r(z) \in \hat{U}[(r \circ s)(x)] \subset (\hat{U} \circ \hat{U})[x] \subset U[x]$. Hence, $h_1(x,t) \in V[\Phi(U[x])]$.

Define homotopies $h_0, h_2 : X \times [0,1] \to Y$ by putting

$$h_0(x,t) := s_1(\tilde{h}(x, 1-t)) \quad \text{and} \quad h_2(x,t) := s_2(\tilde{h}(x,t)), \quad \forall (x,t) \in X \times [0,1].$$

The homotopy $h : X \times [0,1] \to Y$ defined by

$$h(x,t) := \begin{cases} h_0(x,3t), & 0 \leqslant t \leqslant 1/3, \\ h_1(x,3t-1), & 1/3 \leqslant t \leqslant 2/3, \\ h_2(x,3t-2), & 2/3 \leqslant t \leqslant 1 \end{cases}$$

is continuous and joins s_1 and s_2. For every $t \in [0,1]$, $h(.,t) \in \mathbf{a}(\Phi; U', V')$. Property (H) is thus satisfied by Φ. □

Since spaces in AANR($\mathscr{K}$) ($\mathrm{A_H}$ANR($\mathscr{K}$) respectively) are ω-dominated (resp. ω-H-dominated) by finite polyhedra (see Corollary 2.12), we have:

Corollary 3.7. *Assume that X is an* AANR($\mathscr{K}$) (*respectively* $\mathrm{A_H}$ANR($\mathscr{K}$)) *for the class $\mathscr{K}$ of compact spaces and that $Y \in$* NES($\mathscr{K}$). *Then*

$$\mathbf{D}(X,Y) \subset \mathbf{A}(X,Y) \quad (\mathbf{D}(X,Y) \subset \mathbf{A}_H(X,Y), \textit{ respectively}).$$

Inclusion (14) in Appendix A implies the known result:

Corollary 3.8 (Gorniewicz et al. [18]; see Mas Colell [28] for the contractible case)**.** *Assume that X, Y are* ANRs *with X compact. Then,*

$$\mathbf{C}^\infty(X,Y) \subset \mathbf{D}(X,Y) \subset \mathscr{A}_H(X,Y).$$

Remark 3.9. It is an open question as to whether the inclusion above holds when the domain X is a locally finite polyhedron (and consequently for a space dominated by locally finite polyhedra such as arbitrary ANRs) or not. Such spaces are paracompact and Example 3.2(i) and (ii) suggest that the inclusion should be true with mere paracompactness.

An infinite product $X = \prod_{i\in I} X_i$ of compact ANRs X_i, $i \in I$, is a compact space which is not necessarily an ANR. But the finite products $X_J = \prod_{i\in J} X_i$, $J \subset I$ finite, are compact ANRs, hence $\mathbf{D}(X_J, Y) \subset \mathbf{A}(X_J, Y)$. Proposition 2.7, and Theorem 3.6 imply:

Corollary 3.10. *Let $\mathscr{K}$ be the class of compact spaces and let Y be a uniform space. If $X = \prod_{i\in I} X_i$ is an infinite product of compact* ANR*s, then*

$$\mathbf{D}(X, Y) \subset \mathbf{A}(X, Y).$$

Corollary 3.11. *Assume that X is a compact subset of a metrizable locally $\mathscr{C}$-convex space E and that Y is a $\mathscr{C}$-convex subset of a metrizable locally $\mathscr{C}$-convex space F where $\mathscr{C}$ is a c-convexity or a B'-convexity. Then*

$$\mathbf{C}(X, Y) \subset \mathbf{A}_H(X, Y).$$

Proof. We know by Example 3.2(ii) that $\mathbf{C}(X, Y) \subset \mathbf{A}(X, Y)$. According to Proposition 2.4, $E \in \mathscr{A}_H(\mathscr{K}; \mathscr{P})$ where $\mathscr{K}$ is the class of compact spaces and $\mathscr{P}$ is the class of finite polyhedra. Since $X \in \mathscr{K}$, the definition of the class $\mathscr{A}_H(\mathscr{K}; \mathscr{P})$ implies that $X \in D_H(\mathscr{P})$. Moreover, the set Y as well as all values of any map $\Phi \in \mathbf{C}(X, Y)$ are absolute retracts, hence contractible spaces. So, in this case, $\mathbf{C}(X, Y) \subset \mathbf{C}^\infty(X, Y)$. Theorem 3.6 ends the proof. □

4. Fixed point theorems

We recall some basic notions first.

- Given a set X and a set-valued map $\Phi : X \rightrightarrows X$, $\mathrm{Fix}(\Phi) := \{x \in X: x \in \Phi(x)\}$ is the set of all fixed points of Φ.
- Given a set X, a collection $\omega \subset 2^X$, and a set-valued map $\Phi : X \rightrightarrows X$, an element $x \in X$ is said to be a *ω-fixed point* for Φ if both $\{x\}$ and $\Phi(x)$ intersect a common member of ω.
- Let $\mathbf{M}$ be an abstract class of set-valued maps and let X be a space. Following [19], we say that X is a *fixed point space for the class* $\mathbf{M}$ if $\mathrm{Fix}(\Phi) \neq \emptyset$ for all $\Phi \in \mathbf{M}(X)$. We write

$$\mathscr{F}_{\mathbf{M}} := \{X: X \text{ is a fixed point space for the class } \mathbf{M}\}.$$

Fixed points for **USCL** set-valued maps are usually obtained as limits of nets of approximate fixed points. The passage from approximate fixed points to fixed points is provided by the:

Lemma 4.1 (Ben-El-Mechaiekh and Deguire [3]). *Let X be a regular space and $\Phi \in$ **USCL**(X). Assume that there exists a cofinal family $\{\omega\}$ in $\mathrm{Cov}_X(\overline{\Phi(X)})$ such that Φ has a ω-fixed point for all $\omega \in \{\omega\}$. Then Φ has a fixed point.*

Note that in the metrizable setting, the existence of a cofinal family of open covers for a compact set is guaranteed by the Lebesgue number lemma. In a general (i.e., nonnecessary metrizable) uniform space X – which is always completely regular – open covers of a compact subset A admit refinements of the form $\{U[x]\colon x \in A\}$, where U is a member of the uniformity. Thus, such refinements form a cofinal family of open covers. So, in proving the existence of fixed points in uniform spaces for **USCL** compact maps, it suffices to prove the existence of approximate fixed points.

The Himmelberg fixed point theorem on convex subsets of locally convex topological vector spaces was extended to the class $\mathbf{A} \cap \mathbf{CL}$ of approachable closed-valued set-valued maps by the author as follows.

Proposition 4.2 (Ben-El-Mechaiekh [1]). *If X is a nonempty convex subset of a locally convex topological vector space and $\mathscr{K}$ is the class of compact spaces, then $X \in \mathscr{F}_{(\mathbf{A}\cap\mathbf{CL})^{\mathscr{K}}}$, i.e., every compact approachable set-valued map with closed values from X into itself has a fixed point.*

Our interest in the passage $\mathscr{P} \to \mathscr{A}(\mathscr{K};\mathscr{P})$ is not only in the preservation, under very mild assumptions on the spaces and maps involved, of the fixed point property but also in the shifting of compactness from domains to maps. More precisely:

Theorem 4.3. *Let $\mathbf{M}$ be an abstract class of set-valued maps and let $\mathscr{K},\mathscr{P}$ be two classes of topological spaces such that*:

(i) $\mathbf{c} \subset \mathbf{M}$;

(ii) $(\Phi \in \mathbf{M}_{\mathrm{c}}(X,Y))$ *and* $\overline{\Phi(X)} \subset O \underset{\text{open}}{\subset} Y) \Rightarrow (\Phi \cap O \in \mathbf{M}_{\mathrm{c}}(X,O))$;

(iii) *each space in $\mathscr{A}(\mathscr{K};\mathscr{P})$ is regular.*

Then

$$\mathscr{P} \subset \mathscr{F}_{\mathbf{M}_{\mathrm{c}}} \Rightarrow \mathscr{A}(\mathscr{K};\mathscr{P}) \subset \mathscr{F}_{\mathbf{CL}\cap(\mathbf{M}_{\mathrm{c}}^{\mathscr{K}})}. \tag{11}$$

Proof. Let $X \in \mathscr{A}(\mathscr{K};\mathscr{P})$, $\Phi \in \mathbf{M}_{\mathrm{c}}^{\mathscr{K}}(X) \cap \mathbf{CL}$ be arbitrary, i.e., Φ is a closed-valued finite composition of $\mathbf{M}$-maps and $\Phi(X) \subseteq K \subseteq X$, $K \in \mathscr{K}$. Let $\omega \in \mathrm{Cov}_X(K)$ be arbitrary. By (1), there exist a cover $\omega' \in \mathrm{Cov}_X(K), \omega' \preccurlyeq \omega$, a space $P \in \mathscr{P}$ and a pair of continuous mappings $\bigcup \omega' \xrightarrow{s} P \xrightarrow{r} X$ such that $r \circ s$ and $\mathrm{id}_{\cup\omega'}$ are ω-near. The diagram

$$\begin{array}{ccccc} & X & \overset{\Phi'}{\rightrightarrows} & \bigcup \omega' & \overset{s}{\rightarrow} & P \\ r\circ s\circ\Phi' \upuparrows & & & \swarrow r & & \upuparrows s\circ\Phi'\circ r \\ & X & \underset{\Phi'}{\rightrightarrows} & \bigcup \omega' & \underset{s}{\rightarrow} & P \end{array}$$

where $\Phi'(x) := \Phi(x) \cap \bigcup \omega'$, commutes. By (i) and (ii), s,r, and Φ' belong to $\mathbf{M}$, so $s \circ \Phi' \circ r \in \mathbf{M}_{\mathrm{c}}$. Hence, $s \circ \Phi' \circ r$ has a fixed point. It follows that $r \circ s \circ \Phi'$ also has a fixed point $x_\omega \in r(s(\Phi'(x_\omega)))$. Such a fixed point is a ω-fixed point for Φ. Lemma 4.1 ends the proof. □

Remark 4.4. (1) Condition (ii) is superfluous for many particular examples of set-valued maps. For instance, if $\Phi \in \mathbf{K}$, the class of upper semicontinuous maps with convex values (see Section 3 for the definition), restricting the codomain from Y to an open subset of Y containing the range of Φ

does not change the nature of the map Φ. However, in some other cases (e.g., for the class $\mathbf{D}$), relevant features depend on the way the values $\Phi(x)$ of Φ are imbedded in Y rather than on their intrinsic topological properties; thus, restricting the codomain without altering the values themselves may disqualify a map from belonging to a certain class.

(2) Theorem 4.3 implies, in particular, that if a regular topological space is a fixed point space for a class of closed-valued upper semicontinuous maps then so is every retract of the space.

Corollary 4.5. *Let $\mathscr{K}$ be the class of compact spaces and $\mathscr{P}$ the class of convex finite polyhedra. Then,*

$$\mathscr{A}(\mathscr{K};\mathscr{P}) \subset \mathscr{F}_{(\mathbf{A}\cap\mathbf{CL})_c^{\mathscr{K}}},$$

that is, every compact finite composite of approachable maps with closed values of a space $X \in \mathscr{A}(\mathscr{K};\mathscr{P})$ has a fixed point.

Proof. We apply Theorem 4.3 to the class $\mathbf{M}=\mathbf{A}\cap\mathbf{USCO}$. Naturally, every continuous single-valued mapping is approachable (by itself), that is $\mathbf{c}\subset\mathbf{A}\cap\mathbf{USCO}$. From the definition of the class $\mathbf{A}$ (see (5)), one easily verifies that for given uniform spaces X, Y and a map $\Phi \in (\mathbf{A}\cap\mathbf{USCO})(X, Y)$, any open neighborhood O of $\overline{\Phi(X)}$ in Y contains the ranges of (U, V)-approximative selections of Φ for all members U, V of the respective uniformities on X, Y that are small enough. This implies that condition (ii) is always satisfied by the class $\mathbf{A}\cap\mathbf{USCO}$.

Since any finite polyhedron P is a compact space, (6) implies that $(\mathbf{A}\cap\mathbf{USCO})_c(P) = (\mathbf{A}\cap\mathbf{USCO})(P)$.

It remains to verify that every convex finite polyhedra has the fixed point property for u.s.c. closed-valued approachable maps.

But this follows from Proposition 4.2, or by the following elementary direct argument. Consider a convex finite polyhedra P imbedded in a euclidean space E (the topology induced by the euclidean norm on E is uniformizable) and consider a set-valued map $\Phi \in (\mathbf{A}\cap\mathbf{USCO})(P)$. By (5), for any $\varepsilon > 0$, there exists $s \in \mathbf{c}(P)$ such that

$$\forall x \in P, \exists x' \in B_\varepsilon(x), \text{ with } s(x) \in B_\varepsilon(\Phi(x'))$$

The Brouwer fixed point theorem guarantees the existence of a fixed point $x_\varepsilon = s(x_\varepsilon)$ of s in P. To a given sequence $\{\varepsilon_n\}$ of positive real numbers converging to 0, there corresponds a sequence $\{(x_n, y_n)\}$ in $P\times P$, $x_n = x_{\varepsilon_n} = s(x_{\varepsilon_n}) \in B_{\varepsilon_n}(y_n)$, $y_n \in \Phi(B_{\varepsilon_n}(x_n))$. This sequence has a cluster point (x_0, y_0) due to the compactness of P. It is clear that $x_0=s(x_0)=y_0$. Moreover, since a u.s.c. set-valued maps with closed values in a compact space has closed graph and since $\{(x_n, y_n)\}\subset B_{\varepsilon_n}(\mathrm{graph}(\Phi))$ in $P\times P$, it follows that $(x_0, x_0) \in \mathrm{graph}(\Phi)$. This completes the proof of the inclusion $\mathscr{P}\subset\mathscr{F}_{(\mathbf{A}\cap\mathbf{USCO})_c}$ and the proof of our assertion. □

In view of Remark 2.3 and Proposition 2.4, this last result contains, in addition to the classical fixed point theorems of Ky Fan [15] and Himmelberg [20], recent fixed point theorems of the author [1], Deguire and the author [3], Ben-El-Mechaiekh et al. [2], Bielawski [5], Horvath [21], Park [34], Park and Kim [35], Tarafdar [38], and Yuan [41], for $\mathscr{C}$-convex sets (with the various notions of convexity described in Appendix A).

Corollary 4.6. *Assume that X is a non-empty $\mathscr{C}$-convex subset of a locally $\mathscr{C}$-convex space E where $\mathscr{C}$ is*

(i) *a linear convexity structure and E a locally convex space, or*
(ii) *a convexity structure in the sense of Horvath, or*
(ii) *a B-convexity of Bielawski, or*
(iii) *an L-convexity in the sense of Definition* A.2.

Assume also that Φ is a compact map belonging to any one of the classes:
(iv) $\mathbf{C}_c(X)$, *or*
(v) $\mathbf{D}_c(X)$.
Then Φ has a fixed point.

Theorem 4.7. *Let $\mathscr{K}$ and $\mathscr{P}$ be the classes of compact spaces and finite polyhedra respectively. If $X \in \mathscr{A}_H(\mathscr{K};\mathscr{P})$ is compact and $\mathscr{E}(X) \neq 0$, then $X \in \mathscr{F}_{\mathbf{A}_c \cap \mathbf{CL}}$, that is, every compact closed-valued finite composition of approachable set-valued maps from X into itself has a fixed point.*

Proof. Let $\Phi \in (\mathbf{A}_c \cap \mathbf{CL})(X)$. Remember that $\mathscr{K} \cap \mathscr{A}_H(\mathscr{K};\mathscr{P}) \subset D_H(\mathscr{P})$. By Theorem 2.4, for any given $\omega \in \mathrm{Cov}(X)$, there is a finite polyhedron P with nontrivial Euler–Poincaré characteristic that $\omega - H$-dominates X. Such a space has the fixed point property for finite compositions of approachable maps with closed values. This follows from the fact that for a compact space K, we have the equivalence ($K \in \mathscr{F}_{\mathbf{C}} \Leftrightarrow K \in \mathscr{F}_{\mathbf{A}_c \cap \mathbf{CL}}$) (see [3, Proposition 7.1]), and from the Lefschetz theorem for finite polyhedra: $P \in \mathscr{F}_{\mathbf{C}}$. This implies the existence of an ω-fixed point for Φ. Lemma 4.1 ends the proof. □

Theorem 2.11 and Proposition A.10 imply:

Corollary 4.8. *Let $\mathscr{K}$ be the class of compact spaces and let $X \in$* $\mathrm{A_H ANR}(\mathscr{K})$, *Any set-valued map $\Phi \in \mathbf{D}_c(X)$ has a fixed point provided the Euler–Poincaré characteristic $\mathscr{E}(X) \neq 0$.*

Appendix A

In an effort to make this paper self-contained, we include a brief description of some topological notions of convexity that appeared in the literature and on the notions of retracts and neighborhood retracts. The reader is referred to [5,2,21,35] for abstract convexities, and to [6,10,17] for extensive expositions on the theory of retracts. We start with the natural concept of contractibility.

A.1. Contractibility and trivial shape

A space X is *contractible* (in itself) if there exists $h \in \mathbf{c}(X \times [0,1], X)$ such that $h(x,0) = x$ and $h(x,1) = \bar{x}$ where $\bar{x}$ is a given point in X.

Clearly, every star-shaped set is contractible. More particularly, every convex set is contractible. It often happens that a given topological property can be expressed as or implies an extension property which becomes the attractive technical feature of the initial property. In this instance, contractibility implies n-connectedness for all n. More precisely, if for a given positive integer n, Δ^n denotes the standard n-simplex whose vertices $\{e_0,\dots,e_n\}$ form a canonical basis for $\mathbb{R}^{n+1}$and $\partial\Delta^n$ is the boundary of Δ^n, a space X is *n-connected* if and only if $\forall f \in \mathbf{c}(\partial\Delta^n,X), \exists \hat{f} \in \mathbf{c}(\Delta^n,X)$ such that the diagram

$$\begin{tikzcd} & X & \\ \partial\Delta^n \arrow[ur, "f"] \arrow[rr, hook, "i"'] & & \Delta^n \arrow[ul, "\hat{f}"'] \end{tikzcd}$$

commutes. X is *infinitely connected* (C^∞ for short), if X is n-connected for each positive integer n.

Every contractible space is C^∞. Indeed, let $\bar{z}$ be the barycenter of the n-simplex Δ^n and write every $z \in \Delta^n \setminus \{\bar{z}\}$ in polar coordinates form $z = t(z)\bar{z} + (1 - t(z))v(z)$ where $v(z)$ is the radial projection of z onto $\partial\Delta^n$, $t(z) \in [0,1)$. Let $h \in \mathbf{c}(X \times [0,1],X)$ be the homotopy contracting X onto a given point $\bar{x} \in X$ and let $f \in \mathbf{c}(\partial\Delta^n,X)$ be given. One readily verifies that the mapping

$$\hat{f}(z) := \begin{cases} h((f \circ v)(z), t(z)) & \text{if } \ z \neq \bar{z}, \\ \bar{x} & \text{if } \ z = \bar{z} \end{cases}$$

is a continuous extension of f.

We recall next a concept of proximal contractibility important in topology.

Definition A.1 (Van Mill [40]). A subspace A of a space X is said to have trivial shape in X if A is contractible in each of its neighborhoods in X.

Obviously, if $X \supset A$ is contractible, then it has trivial shape in X. However, note that the set

$$\Gamma := \left\{ \left(t, \sin\left(\frac{1}{t}\right)\right) \in \mathbb{R}^2\colon\ 0 < t \leqslant 1 \right\} \cup \{(0,v)\colon\ -1 \leqslant v \leqslant 1\}$$

is not contractible but has trivial shape in $\mathbb{R}^2$. Indeed, $\Gamma := \bigcap_{n\geqslant 1} \Gamma_n$ where $\{\Gamma_n\}_{n\geqslant 1}$ is the decreasing sequence of compact contractible sets:

$$\Gamma_n := \left\{ \left(t, \sin\left(\frac{1}{t}\right)\right) \in \mathbb{R}^2\colon\ 1/n \leqslant t \leqslant \alpha \right\} \cup \{[0,1/n] \times [-1,1]\}, \quad n \geqslant 1.$$

Such an intersection, known as an *R_δ-set*, has trivial shape in its underlying space.

The Borsuk homotopy extension theorem (see [40]) implies that if X is an ANR (see definition below) and if $X \supset A$ has trivial shape in X then, for each open neighborhood U of A in X, there exists an open neighborhood V of A in X contained and contractible in U.

A non-metrizable version of this proximal contractibility follows from the proof of Proposition A.15 below. Assume that $X \in \mathrm{NES}(\mathscr{K})$ where $\mathscr{K}$ is the class of compact spaces and assume that A is a compact subspace of X. Let U be an open neighborhood of A in X and let $h : A \times [0,1] \to U$ be a homotopy joining $\mathrm{id}_X|A = \mathrm{id}_A$ and a constant mapping $\bar{a} \in U$. Since U is itself an NES($\mathscr{K}$) space, we show that the compact homotopy h extends to a homotopy $\hat{h} : V \times [0,1] \to U$ defined on open neighborhood V of A in U and joining id_V to the constant $\bar{a}$.

Consequently, in both cases, ($A \subset X \in$ ANR) or ($A \underset{\text{compact}}{\subset} X \in \mathrm{NES}(\mathscr{K})$), for every positive integer n, every mapping in $\mathbf{c}(\partial\Delta^n, V)$ extends continuously to a mapping in $\mathbf{c}(\Delta^n, U)$. A subspace A with this proximal contractibility is known to be *∞-proximally connected in X* in the sense of Dugundji [13].

A.2. Topological convexities

A *convexity structure* on a set X is a collection $\mathscr{C}$ of subsets of X stable for arbitrary intersections and containing $\emptyset$ and X itself. The pair $(X,\mathscr{C})$ is referred to as a $\mathscr{C}$-*space* and a subset A of X is $\mathscr{C}$-*convex* if $A \in \mathscr{C}$. A convex hull operator is defined by $\mathscr{C}\text{-co}(A) := \bigcap\{B \in \mathscr{C}\colon A \subset B\}$.

There has been a growing interest in abstract convexities in recent years. Some are related to metrizability and the existence of minimal sets as well as to fixed point theory for non-expansive mappings, some to integrability, and others are purely topological and relate to fixed point for trees, etc.

The following abstract topological convexity was defined in [23]. It encompasses a variety of topological convexities studied in the literature.

Definition A.2. An L-structure on a topological space Y is a set-valued map $\Gamma : \langle Y\rangle \rightrightarrows Y$ defined on the family $\langle Y\rangle$ of all nonempty finite subsets of Y verifying

$$\begin{aligned} &\forall A = \{y_0,\ldots,y_n\} \in \langle Y\rangle, \exists f^A \in \mathbf{c}(\Delta^n, \Gamma(A)) \text{ such that} \\ &\qquad f^A(\Delta_J) \subset \Gamma(\{y_i\colon i \in J\}), \quad \forall J \subset \{0,\ldots,n\}, \end{aligned} \tag{12}$$

where for $J \subset \{0,\ldots,n\}$, $\Delta_J = \text{conv}(\{e_i\colon i \in J\})$.

The pair (Y,Γ) is called an L-space and a subset $C \subseteq Y$ is said to be convex for the L-structure if and only if $\Gamma(A) \subset C$, $\forall A \in \langle C\rangle$.

The collection of all convex subsets of Y form a convexity structure $\mathscr{C}$ on Y called an L-*convexity*. The concept of an L-space is actually a refinement of that of a c-space of Horvath.

Example A.3 (Horvath [21]). A pair $(Y, \Gamma : \langle Y\rangle \rightrightarrows Y)$ is said to be a c-space if Γ is isotone ($A \subset B \Rightarrow \Gamma(A) \subset \Gamma(B)$) and has contractible (in fact, C^∞) values.

The reader is referred to [21] for a list of interesting examples of c-spaces. For instance:

- Let Y be topological space and $\alpha : Y \times Y \times [0,1] \to Y$ be a mapping such that

$$\forall (x,y) \in Y \times Y, \quad \alpha(x,y,0) = \alpha(y,x,1) = x.$$

- A subset B of Y is said to be an α-set if $\alpha(B \times B \times [0,1]) \subseteq B$. For any $A \in \langle Y\rangle$, let $\Gamma(A) := \{B\colon A \subseteq B$ and B is an α-set$\}$ and suppose that there exists $a_0 \in \Gamma(A)$ such that the mapping $(y,t) \to \alpha(y,a_0,t)$ is continuous on $\Gamma(A) \times [0,1]$. Then (Y,Γ) is a c-space.
- In particular, given a contractible semigroup $(G,*)$ with unit e and contractibility mapping θ: $G \times [0,1] \to G, \theta(x,1) = x$ and $\theta(x,0) = e, x \in G$, let $\alpha(x,y,t) := \theta(x,1-t) * \theta(y,t)$, $x,y \in G, t \in [0,1]$. Then, (G,Γ) is a c-space.
- Let $Y := L^1_\mu(X,E)$ be the space of μ-integrable functions from a measurable space X with non-atomic probability measure μ into a Banach space E. Given $A := \{g_1,\ldots,g_n\} \subset Y$, let:

$$\Gamma(A) := \left\{ \sum_{i=0}^{n} \mathbf{1}_{X_i} g_i\colon \{X_i\} \text{ being a partition of } X \text{ into } \mu\text{-measurable sets} \right\}.$$

The pair (Y,Γ) is a c-space.

Example A.4. Generalized convex spaces of Park and Kim [35] are L-spaces where, in addition to (4), the L-structure Γ is isotone. Recently, Park removed the isotony condition in his definition.

Simplicial convexities also give rise to L-structures (see [2]).

Example A.5. Let us call a B'-simplicial convexity on a topological space Y any family of continuous functions $\phi^{[x_0,\ldots,x_n]} : \Delta^n \to Y$, defined for each positive integer n and each finite subset $\{y_0,\ldots,y_n\} \subset Y$ and satisfying, $\forall n \geqslant 1, \forall \{y_0,\ldots,y_n\} \subset Y$,

$$\lambda = \sum_{k=0}^{p} \lambda_{i_k} e_{i_k} \Rightarrow \phi^{[y_{i_0},\ldots,y_{i_p}]}(\lambda) = \phi^{[y_0,\ldots,y_n]}\left(\sum_{k=0}^{p} \lambda_{i_k} e_k\right).$$

The set-valued map $\Gamma : \langle Y \rangle \rightrightarrows Y$ defined by

$$\Gamma(A) := \{\phi^{[y_0,\ldots,y_n]}(\lambda) \colon \{y_0,\ldots,y_n\} \subset A,\ \lambda \in \Delta^n\}, A \in \langle Y \rangle,$$

defines an L-structure on Y.

(Any B-simplicial convexity in the sense of [5] is a B'-simplicial convexity with the additional property: $\phi^{[y]}(1) = y$.)

A convexity structure deriving from a B'-simplicial convexity is called a B'-convexity.

It is proven in [2] that every c-space in the sense of Horvath admits a B'-simplicial convexity and is therefore an L-space.

Definition A.6. A uniform space $(Y, \mathscr{V})$ with a convexity structure $\mathscr{C}$ is said to be locally $\mathscr{C}$-convex if

$$\forall V \in \mathscr{V}, \quad A \in \mathscr{C} \Rightarrow V[A] \in \mathscr{C}. \tag{13}$$

This is a more general definition than that of an lc-space of Horvath [21], which is a c-space (Y, Γ) with a basis $\{V_i\}_{i \in I}$ for a compatible uniformity such that $\forall i \in I$, the V_i-neighborhood $V_i[A]$ of a $\mathscr{C}$-convex set A, is also a $\mathscr{C}$-convex set and open balls $V_i[y]$ are $\mathscr{C}$-convex (singletons, in our definition, are not assumed to be $\mathscr{C}$-convex).

A.3. Retracts and neighborhood retracts

A.3.1. ARs and ANRs

Definition A.7. (i) A closed subspace A of a topological space X is a neighborhood retract of X if there exist an open neighborhood V of A in X and a mapping $r \in \mathbf{c}(V, A)$ – such that the diagram

$$\begin{tikzcd} & A & \\ A \arrow[ur, "\mathrm{id}_A"] \arrow[rr, hook, "i"'] & & V \arrow[ul, "r"'] \end{tikzcd}$$

commutes. If $V = X$, A is simply said to be a retract of X.

(ii) A topological space A is an absolute (neighborhood) retract for a class $\mathscr{Q}$ of topological spaces, written $A \in \mathrm{AR}(\mathscr{Q}), (\mathrm{ANR}(\mathscr{Q})$ respectively), if and only if:
(a) $A \in \mathscr{Q}$, and
(b) for every closed imbedding h of A in a space $X \in \mathscr{Q}$, $h(A)$ is a (neighborhood) retract of X.

Obviously, $\mathrm{ANR}(\mathscr{Q})$ contains the class $\mathrm{AR}(\mathscr{Q})$

- If $\mathscr{M} :=$ the class of metric spaces, then $\mathrm{AR}(\mathscr{M})$ is precisely the class AR of *absolute retracts*, and $\mathrm{ANR}(\mathscr{M})$ is simply the class ANR of *absolute neighborhood retracts*.
  - Prototypes of ARs and ANRs are respectively balls and spheres in Euclidean spaces.
  - A theorem of Dugundji asserts that every infinite polyhedron endowed with a metrizable topology is an AR.
  - Every Fréchet manifold is an ANR.
  - The union $C := \bigcup_{i=1}^{n} C_i$, if it is metrizable, of closed convex subsets $C_1, \ldots, C_n$, of a locally convex space E is an ANR.

A.3.2. AANRs and A_H ANRs
We consider now a larger class of neighborhood retracts.

Definition A.8. (i) Let X be a topological space. A closed subset $A \overset{i}{\hookrightarrow} X$ is an approximative neighborhood retract of Y if for any $\omega \in \mathrm{Cov}_X(A)$, there exist an open neighborhood V of A in X and $r \in \mathbf{c}(V, A)$ such that $r \circ i$ and id_A in the following diagram

$$\begin{tikzcd} & A & \\ A \arrow[ur, "\mathrm{id}_A"] \arrow[rr, hook, "i"'] & & V \arrow[ul, "r"'] \end{tikzcd}$$

are ω-near.

(ii) A topological space A is said to be an approximative absolute neighborhood retract for a class of topological spaces $\mathscr{Q}$, written $A \in \mathrm{AANR}(\mathscr{Q})$, if and only if:
(a) $A \in \mathscr{Q}$, and
(b) for every closed imbedding h of A in a space $X \in \mathscr{Q}$, $h(A)$ is an approximative neighborhood retract of X.

The class $\mathrm{A}_H\mathrm{ANR}(\mathscr{Q})$ is defined in a similar way with "ω-near" replaced by "ω-homotopic". Obviously, $\mathrm{AANR}(\mathscr{Q})$ contains $\mathrm{A}_H\mathrm{ANR}(\mathscr{Q})$ and $\mathrm{ANR}(\mathscr{Q})$.

- If $\mathscr{M} :=$ is the class of metric spaces, then $\mathrm{AANR}(\mathscr{M})$ – written AANR for short – is the class of *approximative absolute neighborhood retracts*. One characterizes AANRs as metrizable spaces that are homeomorphic to approximative neighborhood retracts of normed spaces.
  Obviously, ANR $\subseteq$ AANR. This inclusion is strict. Indeed, the set

$$\Gamma := \left\{ \left(x, \sin\left(\frac{1}{x}\right) \right) \in \mathbb{R}^2 \colon 0 < x \leqslant 1 \right\} \cup \{(0, y)\colon -1 \leqslant y \leqslant 1\}$$

  is an AANR. However, because Γ is not locally contractible, it cannot be an ANR.

- If $\mathscr{K}$:= is the class of compact topological spaces, then (see [17])

$$\mathrm{AANR} \cap \mathscr{K} \subset \mathrm{AANR}(\mathscr{K}). \tag{14}$$

A.4. Extension and neighborhood extension spaces

Retracts, neighborhood retracts, and approximative neighborhood retracts can be characterized by extension properties.

Definition A.9. (i) A space X is an extension space for a class of spaces $\mathscr{Q}$, written $X \in \mathrm{ES}(\mathscr{Q})$ if and only if $\forall Y \in \mathscr{Q}$, $\forall K$ closed in Y, $\forall f_0 \in \mathbf{c}(K,X)$, $\exists f \in \mathbf{c}(Y,X)$ such that the diagram

$$\begin{tikzcd} & X & \\ K \arrow[ur, "f_0"] \arrow[rr, hook, "i"'] & & Y \arrow[ul, "f"'] \end{tikzcd}$$

commutes.

(ii) A space X is a neighborhood extension space for a class of spaces $\mathscr{Q}$, written $X \in \mathrm{NES}(\mathscr{Q})$ if and only if $\forall Y \in \mathscr{Q}$, $\forall K$ closed in Y, $\forall f_0 \in \mathbf{c}(K,X)$, $\exists V$ open neighborhood of K in Y, $\exists f \in \mathbf{c}(V,X)$ such that the diagram

$$\begin{tikzcd} & X & \\ K \arrow[ur, "f_0"] \arrow[rr, hook, "i"'] & & V \arrow[ul, "f"'] \end{tikzcd}$$

commutes.

(iii) A space X is an approximative extension space for a class of spaces $\mathscr{Q}$, written $X \in \mathrm{AES}(\mathscr{Q})$ if and only if $\forall \omega \in \mathrm{Cov}(X)$, $\forall Y \in \mathscr{Q}$, $\forall K$ closed in Y, $\forall f_0 \in \mathbf{c}(K,X)$, $\exists f \in \mathbf{c}(Y,X)$, such that $f \circ i$ and f_0 are ω-near.

(iv) A space X is an approximative neighborhood extension space for a class of spaces $\mathscr{Q}$, written $X \in \mathrm{ANES}(\mathscr{Q})$ if and only if $\forall \omega \in Cov(X), \forall Y \in \mathscr{Q}, \forall K$ closed in $Y, \forall f_0 \in \mathbf{c}(K,X), \exists V$ open neighborhood of K in $Y, \exists f \in \mathbf{c}(V,X)$, such that $f \circ i$ and f_0 are ω-near.

Clearly,

$$\mathrm{ES}(\mathscr{Q}) \subset \left\{ \begin{matrix} \mathrm{AES}(\mathscr{Q}) \\ \mathrm{NES}(\mathscr{Q}) \end{matrix} \right\} \subset \mathrm{ANES}(\mathscr{Q}).$$

The classes $\mathrm{A}_H\mathrm{ES}(\mathscr{Q})$ and $\mathrm{A}_H\mathrm{NES}(\mathscr{Q})$ of approximative H-(neighborhood) extension spaces for $\mathscr{Q}$, are defined in a similar way with "ω-near" replaced by "ω-homotopic".

Proposition A.10. *If $\mathscr{Q}$ is a class of normal spaces, then* $\mathrm{AR}(\mathscr{Q}) = \mathscr{Q} \cap \mathrm{ES}(\mathscr{Q})$, $\mathrm{ANR}(\mathscr{Q}) = \mathscr{Q} \cap \mathrm{NES}(\mathscr{Q})$, $\mathrm{AANR}(\mathscr{Q}) = \mathscr{Q} \cap \mathrm{ANES}(\mathscr{Q})$, *and* $\mathrm{A}_H\mathrm{ANR}(\mathscr{Q}) = \mathscr{Q} \cap \mathrm{A}_H\mathrm{NES}(\mathscr{Q})$.

$\mathscr{C}$-convex sets are extension spaces for metric spaces:

Example A.11 (Horvath [22]). Any lc-space E or any c-convex subset of an lc-space is an ES($\mathscr{M}$) where $\mathscr{M}$ is the class of metric spaces. Consequently, any metrizable lc-space E or any c-convex subset of a metrizable lc-space is an AR.

We have the remarkable characterization of ARs:

Example A.12 (Bielawski [5, Theorem 2.1, Corollary 2.2]). Any topological space E equipped with a local B-simplicial convexity is an ES($\mathscr{M}$) where $\mathscr{M}$ is the class of metric spaces. In fact, a metrizable space E is an AR if and only if E can be equipped with a local B-simplicial convexity.

It is well-known that every ANR Y is homeomorphic to a neighborhood retract of a normed space (namely, the Banach space of all bounded continuous real-valued functions on Y; see [11, Theorem 7.1] or [19, Example 2.2]. But normed spaces are ES($\mathscr{K}$) for the class $\mathscr{K}$ of compact spaces and, for any class of spaces $\mathscr{Q}$, neighborhood retracts of an NES($\mathscr{Q}$) are also NES($\mathscr{Q}$). Hence,

Example A.13. ANR $\subset$ NES($\mathscr{K}$) for the class $\mathscr{K}$ of compact spaces.

The fact that ANRs can be imbedded as neighborhood retracts of normed spaces (which are of course locally convex) is used to construct linear homotopies between mappings that are close. More precisely, we have the crucial observation:

Proposition A.14 (Dugundji [11, Lemma 7.2]). *Given a metric space X, an* ANR Y, *and a cover $\omega \in \mathrm{Cov}(Y)$, there exists $\omega' \in \mathrm{Cov}(Y), \omega' \preccurlyeq \omega$, such that any two mappings $f, g : X \to Y$ that are ω'-near are ω-homotopic.*

This property can be extended to nonmetrizable NES($\mathscr{K}$) spaces by using a generalization of the "controlled" Borsuk homotopy extension theorem. We can show:

Proposition A.15. *Compact subspaces of* NES($\mathscr{K}$) *spaces, where $\mathscr{K}$ is the class of compact spaces, are uniformly locally contractible.* [2] *Consequently, for any open cover ω of Z, any two continuous mappings with values in a compact subspace Z of an* NES($\mathscr{K}$) *space that are close enough are ω-homotopic.*

[2] Let Z be a subspace of a topological space Y and assume that Z has a uniform structure $\mathscr{V}$. Z is said to be *ω-uniformly contractible in Y* (ω-ULC in Y, for short) for a given open cover $\omega \in \mathrm{Cov}_Y(Z)$, if there is a member $V \in \mathscr{V}$ and a continuous mapping $\xi : V \times [0,1] \to Y$ such that

$$\forall z, z' \in V,\ \forall t \in [0,1],\ \xi(z,z',0) = \xi(z',z,1) = z,\ \xi(z,z,t) = z,$$
$$\text{and } \xi((z,z') \times [0,1]) \text{ is contained in a member } W \text{ of } \omega.$$

Z is said to be *uniformly locally contractible* in Y (ULC in Y, for short) if it is ω-ULC in Y, for any $\omega \in \mathrm{Cov}_Y(Z)$. If Z is ω-ULC in Y then there exists $V \in \mathscr{V}$ such that any two continuous mappings $f, g : X \to Z$ defined on a space X that are V-near are ω-homotopic.

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